Projective Geometric Algebra powers up Bézier Curves

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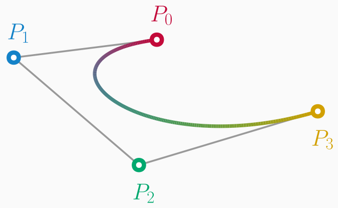
Classical Bézier Curves

Polynomial Bézier (degree d):

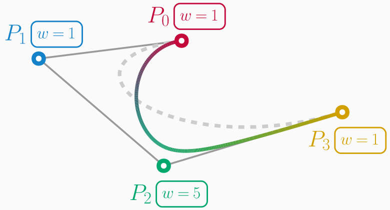
$$P(t) = \sum_{i=0}^{d} B_i^d(t)\, P_i \qquad P_i \in \mathbb{R}^2$$

Bernstein basis $B_i^d(t) = \binom{d}{i} t^i (1-t)^{d-i}$, $t \in [0,1]$.



Rational Bézier (weights $w_i > 0$):

$$P(t) = \frac{\sum_i B_i^d(t)\, w_i\, P_i}{\sum_i B_i^d(t)\, w_i}$$

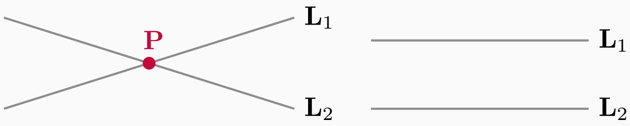


PGA : Plane-based Geometric Algebra

$$\mathbf{e}_0^2 = 0 \quad \mathbf{e}_1^2 = \ldots = \mathbf{e}_n^2 = 1$$
$$\mathbf{e}_i \cdot \mathbf{e}_j = 0, i \neq j$$

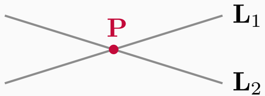
$$\textbf{Hyperplanes : } \mathbf{a} = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$$
$$\textbf{Points : } \mathbf{P} = (w\mathbf{e}_0 + x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n)^\star$$

In 2D: hyperplanes $\rightarrow$ **Lines** : $\mathbf{L} = c\mathbf{e}_0 + a\mathbf{e}_1 + b\mathbf{e}_2$



$$\mathbf{P} = (w\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2)^\star$$

PGA Finite and Ideal



Finite Point

$$\mathbf{P} = (w\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2)^\star$$

$$\|\mathbf{P}\| = \|\mathbf{P}\|_f = \sqrt{\mathbf{P}\,\widetilde{\mathbf{P}}} = |w|$$

Ideal Point

$$\mathbf{P} = (0\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2)^\star$$

$$\|\mathbf{P}\| = \|\mathbf{P}\|_\infty = \|\mathbf{P}^\star\|_f = \sqrt{x^2 + y^2}$$

Adaptative normalization (coordinate-free):

$$\hat{A} = \frac{A}{\|A\|}$$

Bézier Curves in PGA

Projective Bézier Curves in PGA

The **PGA Bézier curve**

$$\boxed{\mathbf{P}(t) = \sum_{i=0}^{d} B_i^d(t)\,\lambda_i\,\hat{\mathbf{P}}_i}$$

$$= \sum_{i=0}^{d} B_i^d(t)\,\lambda_i\,(\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2)^\star$$

$$= \underbrace{\sum_{i=0}^{d} B_i^d(t)\,\lambda_i \mathbf{e}_0^\star}_{w\mathbf{e}_0^\star} + \sum_{i=0}^{d} B_i^d(t)\,\lambda_i (x\mathbf{e}_1 + y\mathbf{e}_2)^\star$$

Projective normalization (coordinate-free):

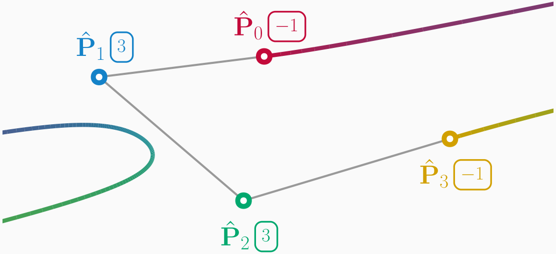
$$\hat{\mathbf{P}}(t) = \frac{\mathbf{P}(t)}{\|\mathbf{P}(t)\|} = \frac{\sum_{i=0}^{d} B_i^d(t)\,\lambda_i (\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2)^\star}{|\sum_{i=0}^{d} B_i^d(t)\,\lambda_i|} \underbrace{=}_{\lambda_i > 0} \frac{\sum_{i=0}^{d} B_i^d(t)\,\lambda_i \mathbf{P}_i}{\sum_{i=0}^{d} B_i^d(t)\,\lambda_i}$$

Negative Weights: $\lambda_i \in \mathbb{R}^*$

If $\sum_i B_i^d(t)\lambda_i = 0 \leftrightarrow w(t) = 0$ for some t : $\mathbf{P}(t)$ is just an **ideal point** !

$$\hat{\mathbf{P}}(t) = \frac{\mathbf{P}(t)}{\|\mathbf{P}(t)\|}$$

$$\|\mathbf{P}(t)\| = \begin{cases} \|\mathbf{P}(t)\|_f & \text{if } \mathbf{P}(t) \text{ is finite,} \\ \|\mathbf{P}(t)\|_\infty & \text{if } \mathbf{P}(t) \text{ is ideal} \end{cases}$$



Geometric meaning:

- Negative weight $\rightarrow$ repulsion
- Singularity $\rightarrow$ asymptotic direction

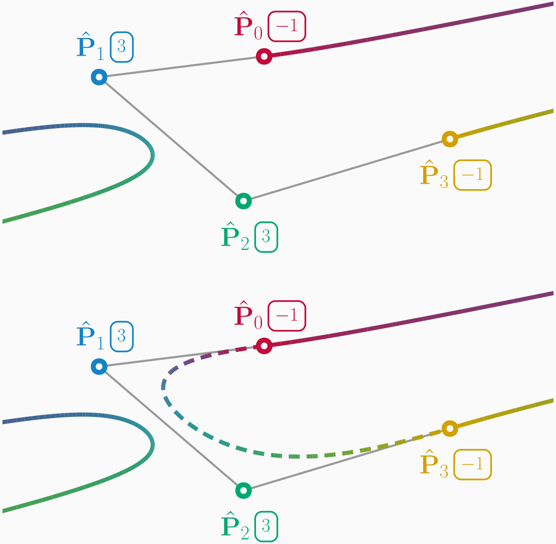
Complementary Curve Section

Endpoint tangents:

$$\mathbf{P}'(0) \propto \frac{\lambda_1}{\lambda_0}(\mathbf{P}_1 - \mathbf{P}_0)$$
$$\mathbf{P}'(1) \propto \frac{\lambda_{d-1}}{\lambda_d}(\mathbf{P}_d - \mathbf{P}_{d-1})$$

Flip endpoint weight signs

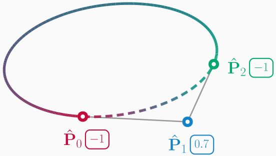
$\Rightarrow$ tangent directions reverse at endpoints



Conics as Quadratic Rational Bézier

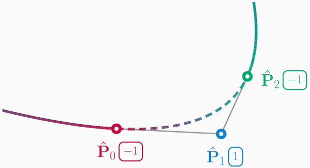
$$\mathbf{P}(t) = \frac{B_0(t)\lambda_0\mathbf{P}_0 + B_1(t)\lambda_1\mathbf{P}_1 + B_2(t)\lambda_2\mathbf{P}_2}{B_0(t)\lambda_0 + B_1(t)\lambda_1 + B_2(t)\lambda_2}$$

$$Q(t) = (1-t)^2\lambda_0 + 2t(1-t)\lambda_1 + t^2\lambda_2 \qquad \Delta = 4(\lambda_1^2 - \lambda_0\lambda_2)$$



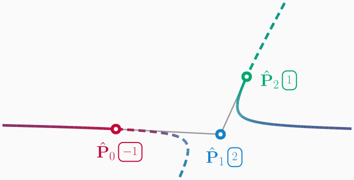
Ellipse

$\Delta < 0$



Parabola

$\Delta = 0$

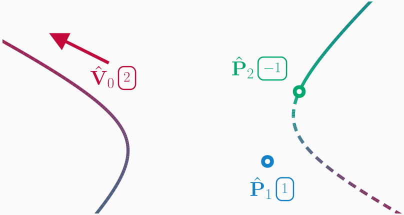


Hyperbola

$\Delta > 0$

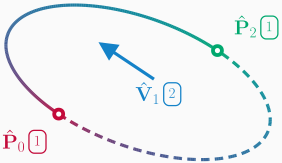
Finite and Ideal Control Points

$$\mathbf{P}(t) = \sum_{i=0}^{d} B_i^d(t)\,\lambda_i\,\hat{\mathbf{P}}_i \qquad \hat{\mathbf{P}}_i = \begin{cases} \dfrac{w\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2}{|w|} & \text{if finite,} \\[2ex] \dfrac{\mathbf{e}_1 + y\mathbf{e}_2}{\sqrt{x^2+y^2}} & \text{if ideal} \end{cases}$$



Endpoint ideal

asymptotic direction

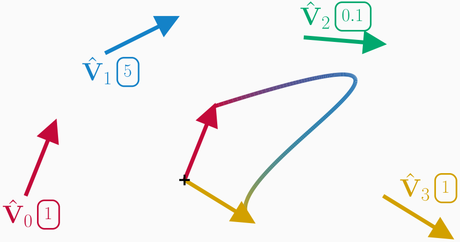


Interior ideal

directional control

Ideal Control Points

$$\mathbf{P}(t) = \sum_{i=0}^{d} B_i^d(t)\, \lambda_i\, \hat{\mathbf{P}}_i \qquad \hat{\mathbf{P}}_i = \frac{x\mathbf{e}_1 + y\mathbf{e}_2}{\sqrt{x^2 + y^2}}$$

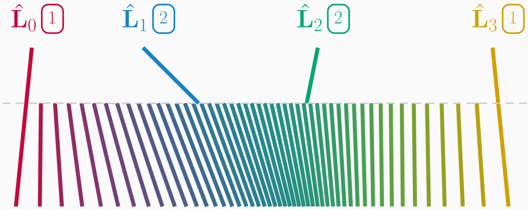


Generalization

Bézier of Lines

$$\mathbf{L} = c\mathbf{e}_0 + a\mathbf{e}_1 + b\mathbf{e}_2 \qquad \hat{\mathbf{L}} = \frac{\mathbf{L}}{\|\mathbf{L}\|}$$

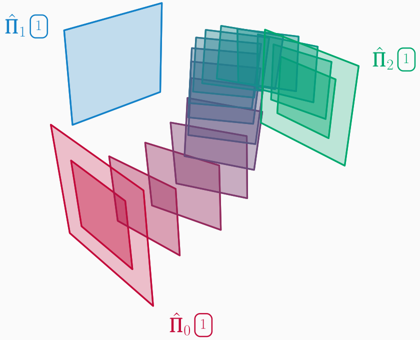
$$\mathbf{L}(t) = \sum_{i=0}^{d} B_i^d(t)\,\lambda_i\,\hat{\mathbf{L}}_i$$



Bézier of Planes

$$\mathbf{\Pi} = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \qquad \hat{\mathbf{\Pi}} = \frac{\mathbf{\Pi}}{\|\mathbf{\Pi}\|}$$

$$\mathbf{\Pi}(t) = \sum_{i=0}^{d} B_i^d(t)\,\lambda_i\,\hat{\mathbf{\Pi}}_i$$

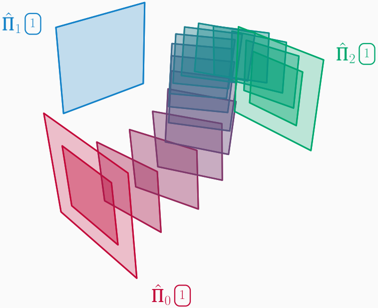


Bézier of Hyperplanes

In n-d PGA : Bézier curve of hyperplanes:

$$\mathbf{h} = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n \qquad \hat{\mathbf{h}} = \frac{\mathbf{h}}{\|\mathbf{h}\|}$$

$$\mathbf{h}(t) = \sum_{i=0}^{d} B_i^d(t)\, \lambda_i\, \hat{\mathbf{h}}_i$$



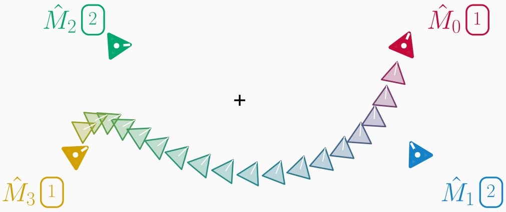
Bézier of Versors: Smooth Rigid Motions

Versors encode rigid transformations :

$$\mathbf{X}(t) = \hat{V}(t)\,\mathbf{X}\,\widetilde{\hat{V}}(t)$$

Bezier of Versors :

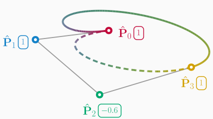
$$V(t) = \sum_{i=0}^{d} B_i^d(t)\,\lambda_i \hat{V}_i$$



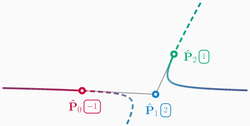
$$\mathbf{X}(t) = \sum_{i=0}^{d} B_i^d(t)\, \lambda_i\, \hat{\mathbf{X}}_i$$

One Equation, many Curves :

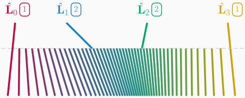
- Points (finite + ideal)
- Negative weights :
  - complementary sections
  - conics
- Lines, planes, hyperplanes
- Versors : rigid-motion animation



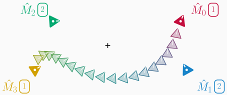
Negative weight



Hyperbola



Bézier of lines

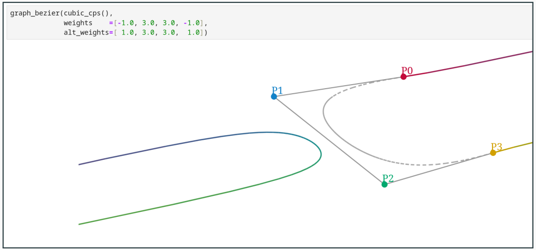


Bézier of motors

Conclusion 2/2

Source code

Interactive notebooks: `https://eharquin.github.io/pga-bezier/`



Future work

- Using complex/versors weights λ_i
- Bézier surfaces and splines in PGA
- Geodesic versor interpolation (exp / log maps, ScLERP)

Questions?